MATH20132 Calculus of Several Variable. 2020-21

Problems 3: The Directional Derivative

1 Define the functions

i. $f : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{x} \mapsto x(x+y)$ and

ii. $g : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{x} \mapsto y(x-y)$.

Find the directional derivatives of f and g at $\mathbf{a} = (1,2)^T$ in the direction $\mathbf{v} = (2,-1)^T/\sqrt{5}$.

2. Find the directional derivative of $f : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{x} \to x^2y$ at $\mathbf{a} = (2,1)^T$ in the direction of the unit vector $\mathbf{v} = (1,-1)^T/\sqrt{2}$.

3. Define the function $h : \mathbb{R}^3 \to \mathbb{R}$, by $\mathbf{x} \to xy + yz + xz$. By verifying the definition, find the directional derivative of h at $\mathbf{a} = (1,2,3)^T$ in the direction of the unit vector $\mathbf{v} = (3,2,1)^T/\sqrt{14}$.

4. Define the function $f : \mathbb{R}^3 \to \mathbb{R}$, by $\mathbf{x} \to xy^2z$. By verifying the definition, find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (1,3,-2)^T$ in the direction of the unit vector $\mathbf{v} = (-1,1,-2)^T/\sqrt{6}$.

5. Define the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathbf{x} \to \begin{pmatrix} xy \\ yz \end{pmatrix},$$

where $\mathbf{x} = (x,y,z)^T$. By verifying the definition, find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (1,3,-2)^T$ in the direction of the unit vector $\mathbf{v} = (-1,1,-2)^T/\sqrt{6}$.

Do **not** look at the component functions separately.

6 Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x\,(x+y) \\ y\,(x-y) \end{pmatrix}.$$

Find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (1,2)^T$ in the direction $\mathbf{v} = (2,-1)^T/\sqrt{5}$.

Hint Notice the difference in wording between this question and the previous one; here I do not ask you to verify the definition.

7 Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}.$$

Find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (2,1)^T$ in the direction $\mathbf{v} = (1,-1)^T/\sqrt{5}$.

8. i. Let $\mathbf{c} \in \mathbb{R}^n$ be fixed. Let $f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$. Show that

$$d_{\mathbf{v}} f(\mathbf{a}) = f(\mathbf{v})$$

for all $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$.

ii. Let $M \in M_{m,n}(\mathbb{R})$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto M\mathbf{x}$. Show that

$$d_{\mathbf{v}} \mathbf{f}(\mathbf{a}) = \mathbf{f}(\mathbf{v})$$

for all $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$.

iii. Can you generalise these results? I.e. of what type of function are $\mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$ and $\mathbf{x} \mapsto M\mathbf{x}$ examples?

9. Assume for the scalar-valued function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ the directional derivative $d_{\mathbf{v}} f(\mathbf{a})$ exists for some $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$. Prove that

$$\lim_{t \to 0} f(\mathbf{a} + t\mathbf{v}) = f(\mathbf{a}).$$

This is yet another example of the principle that if a function is differentiable at a point then it is continuous at that point. There are no new ideas in the proof, look back at previous proofs of differentiable implies continuous.

10. Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $\mathbf{x} \to |\mathbf{x}|$.

i. Prove that f is continuous in any direction at the origin.

ii. Show that in no direction through the origin does f have a directional derivative.

This example illustrates the fact that

continuous in a direction $\not\Rightarrow$ differentiable in that direction.

11. Assume $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathbf{a} \in U$ and we have a unit vector $\mathbf{v} \in \mathbb{R}^n$. Prove that if the directional derivative $d_{\mathbf{v}} f(\mathbf{a})$ exists then so does the directional derivative $d_{-\mathbf{v}} f(\mathbf{a})$ and that it satisfies $d_{-\mathbf{v}} f(\mathbf{a}) = -d_{\mathbf{v}} f(\mathbf{a})$.

12. Using the definition of directional derivative calculate $d_1(x^2y)$ and $d_2(x^2y)$. Hence verify that these directional derivatives are the partial derivatives w.r.t x and y respectively.

13. Find the partial derivatives of the following functions:

i. $f : U \to \mathbb{R}$, $\mathbf{x} \mapsto x \ln(xy)$ where $U = \{\mathbf{x} \in \mathbb{R}^2 : xy > 0\}$;

ii. $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{x} \to (x^2 + 2y^2 + z)^3$;

iii. $f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \to |\mathbf{x}|$ for $\mathbf{x} \neq \mathbf{0}$. What goes wrong when $\mathbf{x} = \mathbf{0}$?

Hint In Part iii write out the definition of $|\mathbf{x}|$.

14. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{x^2y}{x^2+y^2} \quad \text{if} \quad \mathbf{x} \neq \mathbf{0}; \quad f(\mathbf{0}) = 0.$$

This as been previously seen in Question 11iii on Sheet 1.

i. Prove that f is continuous at $\mathbf{0}$.

ii. Find the partial derivatives of f at $\mathbf{0}$. (Hint return to the definition of derivative.)

iii. Prove that $d_{\mathbf{v}} f(\mathbf{0})$ exists for all unit vectors $\mathbf{v}$, and, in fact, equals $f(\mathbf{v})$.

15. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{xy}{x^2+y^2} \quad \text{if} \quad \mathbf{x} \neq \mathbf{0}; \qquad f(\mathbf{0}) = 0.$$

It was shown in Question 11ii on Sheet 1 that f does not have a limit at $\mathbf{0}$ and so is **not** continuous at $\mathbf{x} = \mathbf{0}$.

i. Show that, nonetheless, the partial derivatives of f exist at $\mathbf{0}$.

ii. Prove that for all unit vectors $\mathbf{v} \neq \mathbf{e}_1$ or $\mathbf{e}_2$ the directional derivative $d_{\mathbf{v}} f(\mathbf{0})$ does not exist.

This example illustrates the point that

$$\boxed{\forall i,\ d_i f(\mathbf{a})\,\text{exists} \;\not\Longrightarrow\; \forall \mathbf{v},\ d_{\mathbf{v}} f(\mathbf{a})\,\text{exists}}$$

Additional Questions 3

16. *The Product Rule for directional derivatives*

i. Assume for the scalar-valued functions $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ that the directional derivatives $d_{\mathbf{v}} f(\mathbf{a}), d_{\mathbf{v}} g(\mathbf{a})$ exist for some $\mathbf{a} \in U, \mathbf{v} \in \mathbb{R}^n$. Prove that the directional derivative $d_{\mathbf{v}}(fg)(\mathbf{a})$ exists and satisfies

$$d_{\mathbf{v}}(fg)(\mathbf{a}) = f(\mathbf{a})\, d_{\mathbf{v}} g(\mathbf{a}) + g(\mathbf{a})\, d_{\mathbf{v}} f(\mathbf{a}).$$

ii Use Part i with the result of Question 5 to independently check your answer to Question 4.

Hint in Part i no new ideas are needed; look back to last year at proofs for differentiating products of functions.

17. **Extra questions for practice** From first principles calculate the directional derivatives of the following functions.

i. $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x} \mapsto (x+y, x-y, xy)^T$, at $\mathbf{a} = (2,-1)^T$ in the direction $\mathbf{v} = (1,-2)^T/\sqrt{5}$,

ii. $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2, x \mapsto (x+1, x^2-2)^T$, at $a = 1$ in the direction of $v = -1$,

iii. $h \circ \mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$, with $\mathbf{f}$ as in part i, and $h(\mathbf{x}) = xy^2z$ for $\mathbf{x} \in \mathbb{R}^3$, at $\mathbf{a} = (2,-1)^T$ in the direction $\mathbf{v} = (1,-2)^T/\sqrt{5}$,

iv. $\mathbf{f} \circ \mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ at $a = 1$ in the direction of $v = -1$.

18. Some important functions from the course are

- the projection functions $p^i : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto x^i$;
- the product function $p : \mathbb{R}^2 \mapsto \mathbb{R}$, $\mathbf{x} = (x,y)^T \mapsto xy$ and
- the quotient function $q : \mathbb{R} \times \mathbb{R}^\dagger \to \mathbb{R}$, $\mathbf{x} = (x,y)^T \mapsto x/y$.

Find $d_{\mathbf{v}} p^i(\mathbf{a})$; $d_{\mathbf{v}} p(\mathbf{a})$ for $\mathbf{a}, \mathbf{v} \in \mathbb{R}^2$ and $d_{\mathbf{v}} q(\mathbf{a})$ for $\mathbf{a} \in \mathbb{R} \times \mathbb{R}^\dagger$ and $\mathbf{v} \in \mathbb{R}^2$.